

# TAUT SUBMANIFOLDS ARE ALGEBRAIC

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ABSTRACT. We prove that every (compact) taut submanifold in Euclidean space is real algebraic, i.e., is a connected component of a real irreducible algebraic variety in the same ambient space.

## 1. INTRODUCTION

An embedding  $f$  of a compact, connected manifold  $M$  into Euclidean space  $\mathbb{R}^n$  is *taut* if every nondegenerate (Morse) Euclidean distance function,

$$L_p : M \rightarrow \mathbb{R}, \quad L_p(z) = d(f(z), p)^2, \quad p \in \mathbb{R}^n,$$

has  $\beta(M, \mathbb{Z}_2)$  critical points on  $M$ , where  $\beta(M, \mathbb{Z}_2)$  is the sum of the  $\mathbb{Z}_2$ -Betti numbers of  $M$ . That is,  $L_p$  is a perfect Morse function on  $M$ .

A slight variation of Kuiper's observation in [7] gives that tautness can be rephrased by the property that

$$(1.1) \quad H_j(M \cap B, \mathbb{Z}_2) \rightarrow H_j(M, \mathbb{Z}_2)$$

is injective for all closed disks  $B \subset \mathbb{R}^n$  and all  $0 \leq j \leq \dim(M)$ . As a result, tautness is a conformal invariant, so that via stereographic projection we can reformulate the notion of tautness in the sphere  $S^n$  using the spherical distance functions. Another immediate consequence is that if  $B_1 \subset B_2$ , then

$$(1.2) \quad H_j(M \cap B_1) \rightarrow H_j(M \cap B_2)$$

is injective for all  $j$ .

Kuiper in [8] raised the question whether all taut submanifolds in  $\mathbb{R}^n$  are real algebraic. We established in [4] that a taut submanifold in  $\mathbb{R}^n$  is real algebraic in the sense that, it is a connected component of a real irreducible algebraic variety in the same ambient space, provided the submanifold is of dimension no greater than 4.

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In this paper, we prove that all taut submanifolds in  $\mathbb{R}^n$  are real algebraic in the above sense, so that each is a connected component of a real irreducible algebraic variety in the same ambient space. In particular, any taut hypersurface in  $\mathbb{R}^n$  is described as  $p(t) = 0$  by a single irreducible polynomial  $p(t)$  over  $\mathbb{R}^n$ . Moreover, since a tube with a small radius of a taut submanifold in  $\mathbb{R}^n$  is a taut hypersurface [10], which recovers the taut submanifold along its normals (we will see this in (2.4) below), understanding a taut submanifold, in principle, comes down to understanding the hypersurface case defined by a single algebraic equation.

To achieve the goal, on the one hand we continue to explore the property that certain multiplicity sets are of finite ends as studied in [4]. On the other we employ Morse-Bott theory [3] and further real algebraic geometry in conjunction with Ozawa's theorem [9] to obtain, in the hypersurface case, a fine structure of the set where the principal multiplicities are not locally constant. As a byproduct, the crucial local finiteness property that is decisive in [4] for establishing that a taut submanifold is algebraic falls out.

It is more convenient to prove that a taut submanifolds in the sphere is real algebraic, though occasionally we will switch back to Euclidean space when it is more convenient for the argument. Since a spherical distance function  $d_p(q) = \cos^{-1}(p \cdot q)$  has the same critical points as the Euclidean height function  $\ell_p(q) = p \cdot q$ , for  $p, q \in S^n$ , a compact submanifold  $M \subset S^n$  is taut if and only if it is *tight*, i.e., every nondegenerate height function  $\ell_p$  has the total Betti number  $\beta(M, \mathbb{Z}_2)$  of critical points on  $M$ . We will use both  $d_p$  and  $\ell_p$  interchangeably, whichever is more convenient for our argument.

Our proof is based on a fundamental result on taut submanifolds due to Ozawa [9].

**Theorem 1** (Ozawa). *Let  $M$  be a taut submanifold in  $S^n$ , and let  $\ell_p, p \in S^n$ , be a linear height function on  $M$ . Let  $x \in M$  be a critical point of  $\ell_p$ , and let  $S$  be the connected component of the critical set of  $\ell_p$  that contains  $x$ . Then  $S$  is*

- (a) *a smooth compact manifold of dimension equal to the nullity of the Hessian of  $\ell_p$  at  $x$ ;*
- (b) *nondegenerate as a critical manifold;*
- (c) *taut in  $S^n$ .*

In particular,  $\ell_p$  is perfect Morse-Bott [3]. We call such a connected component of a critical set of  $\ell_p$  a *critical submanifold* of  $\ell_p$ .

An important consequence of Ozawa's theorem is the following [5].

**Corollary 2.** *Let  $M$  be a taut submanifold in  $S^n$ . Then given any principal space  $T$  of any shape operator  $S_\zeta$  at any point  $x \in M$ , there exists a submanifold  $S$  (called a curvature surface) through  $x$  whose tangent space at  $x$  is  $T$ . That is,  $M$  is Dupin [10].*

Let us remark on a few important points in the corollary. It is convenient to work in the ambient Euclidean space  $\mathbb{R}^n$ . Let  $\mu$  be the principal value associated with  $T$ . Consider the focal point  $p = x + \zeta/\mu$ . Then the critical submanifold  $S$  of the (Euclidean) distance function  $L_p$  through  $x$  is exactly the desired curvature surface through  $x$ . The unit vector field

$$(1.3) \quad \zeta(y) := \mu(p - y)$$

for  $y \in S$  extends  $\zeta$  at  $x$  and is normal to and parallel along  $S$ . The  $(n-1)$ -sphere of radius  $1/\mu$  centered at  $p$  is called the *curvature sphere* of  $Z$ .

## 2. THE PROOF

We do an inductive argument on the following statement:

$\mathcal{S}(n)$ : All taut submanifolds in  $S^n$  are real algebraic.

The statement is true for  $n = 1$  since a 0-dimensional taut submanifold is a point. Assuming the statement is true for all  $k \leq n - 1$ .

We first handle the case when  $M$  is a hypersurface. Fix a unit normal field  $\mathbf{n}$  over  $M$  once and for all. We label the principal curvatures of  $M$  by  $\lambda_1 \leq \dots \leq \lambda_{n-1}$ , which are Lipschitz-continuous functions on  $M$  because the principal curvature functions on the linear space  $\mathcal{L}$  of all symmetric matrices are Lipschitz-continuous by general matrix theory [1, p. 64], and the Hessian of  $M$  is a smooth function from  $M$  into  $\mathcal{L}$ . Let  $\lambda_j = \cot(t_j)$  for  $0 < t_j < \pi$ . We have the Lipschitz-continuous focal maps

$$f_j(x) = \cos(t_j)x + \sin(t_j)\mathbf{n}.$$

In fact, the  $l$ th focal point  $f_l(x)$  along  $\mathbf{n}$  emanating from  $x$  is antipodally symmetric to the  $(n-l)$ th focal point along  $-\mathbf{n}$  emanating from  $x$ . The spherical distance functions  $d_{f_l(x)}$  tracing backward following  $-\mathbf{n}$  thus assumes the same critical point  $x$  as the distance function  $d_{-f_l(x)}$  tracing backward following  $\mathbf{n}$ ; thus we may just consider the former case without loss of generality. Accordingly, we refer to a focal point  $p$  as being  $f_j(x)$  for some  $x$  and  $j$ .

By the inductive hypothesis,  $Z$  must be algebraic since  $Z$  lies in its curvature sphere by Corollary 2.

As mentioned earlier, we can regard  $M \subset S^n$  as being tight. Suppose  $Z$  is a critical submanifold of  $M$  cut out by the height function  $\ell_p$ ; assume  $\ell_p(Z) = 0$  without loss of generality. Let  $W \subset M$  be a tubular neighborhood of  $Z$  so small that  $\ell_p^{-1}(0)$  is the only critical set of  $\ell_p$  in  $W$ . (We will call such a  $W$  a *neck* around  $Z$ .)

Let us slightly perturb  $\ell_p$  by a linear function  $g$  with small coefficients such that  $g$  is not a multiple of  $\ell_p$  (otherwise  $\ell_p + g$  is just  $\ell_p$  in essence). Then  $\ell_p + g = \ell_q$  for some  $q$  close to  $p$ .  $Z$  is not a critical submanifold of  $\ell_p + g$ , or equivalently, of  $g$  since  $q \neq p$ .

Since  $Z$  is taut by Ozawa's theorem, in general the height function  $g$  cuts  $Z$  in several critical submanifolds  $Z_1, \dots, Z_l$ ; without loss of generality, we assume these critical submanifolds of  $Z$  correspond to different critical values of  $g$ . It suffices to consider  $Z_1$ , for instance. Assume the codimension of  $Z_1$  in  $Z$  is  $t$  and the dimension of  $Z$  is  $s$ . Let us parametrize  $W$  by  $v_1, \dots, v_t, v_{t+1}, \dots, v_s, u_1, \dots, u_{n-1-s}$  around 0, where  $v_{t+1}, \dots, v_s$  parametrize  $Z_1$ ,  $v_1, \dots, v_s$  parametrize a neck around  $Z_1$  in  $Z$ , and lastly the variables  $v_1, \dots, v_s, u_1, \dots, u_{n-1-s}$  parametrize the neck  $W$  of dimension  $n-1$ , which is the dimension of  $M$ , around  $Z$ . It is understood that 0 in the coordinate system corresponds to a point on  $Z_1$ . As in [9], we can assume

$$\begin{aligned}\ell_p &= \sum_{j=1}^{n-1-s} \alpha_j u_j^2 + O(3), \\ g &= h(u) + \sum_{i=1}^t \beta_i v_i^2 + O(3),\end{aligned}$$

with

$$h(u) = \sum_{j=1}^{n-1-s} a_j u_j + \sum_{j,k=1}^{n-1-s} b_{jk} u_j u_k$$

for some small coefficients  $a_i$  and  $b_{jk}$ , where  $\alpha_j$  and  $\beta_i$  are all nonzero constants. Note that the cross  $uv$ -terms can always be canceled by an appropriate linear change of coordinates. Moreover, there are no  $v_{t+1}, \dots, v_s$  present in  $g$  because when we set the  $u$ -variables equal to zero,  $Z_1$  parametrized by  $v_{t+1}, \dots, v_s$  is a critical submanifold of  $g$  over  $Z$ . Differentiating and setting the derivatives equal to zero, we obtain

$$0 = \partial(\ell_p + g)/\partial u_j = a_j + 2\alpha_j u_j + 2 \sum_{l=1}^{n-1-s} b_{jl} u_l + O(2) := F_j$$

for  $1 \leq j \leq n-1-s$ , and

$$0 = \partial(\ell_p + g)/\partial v_i = 2\beta_i v_i + O(2) := G_i$$

for  $1 \leq i \leq t$ .

Since  $a_i$  and  $b_{jk}$  are small quantities, we know

$$\partial(F_1, \dots, F_{k-s})/\partial(u_1, \dots, u_{n-1-s}) \neq 0$$

at  $u_1 = \dots = u_{n-1-s} = 0$ . Therefore, the implicit function theorem implies that  $u_1, \dots, u_{n-1-s}$  are all functions of  $v_1, \dots, v_s$ . Likewise, since all  $\beta_i$  are nonzero, we can in turn solve  $v_1, \dots, v_t$  in terms of  $v_{t+1}, \dots, v_s$ , the coordinates of  $Z_1$ . The critical set is thus a graph over  $Z_1$ . Hence we have the following.

**Proposition 3.** *Consider a neck  $W$  around a critical submanifold  $Z$  that is cut out by  $\ell_p$  in  $M$ . Let  $N_1, \dots, N_l$  be necks around the critical submanifolds  $Z_1, \dots, Z_l$  cut out by  $g$  in  $Z$ , respectively. Set up a finite number of aforementioned coordinate charts and let*

$$\pi : (v_1, \dots, v_s, u_1, \dots, u_{n-1-s}) \mapsto (v_1, \dots, v_s).$$

*be the projection. Then the critical set of  $\ell_p + g$  in  $\pi^{-1}(N_i)$  is a graph over  $Z_i$ .*

On the other hand, at a point  $x \in Z$  away from  $Z_1, \dots, Z_l$ , we can still parametrize  $W$  around  $x$  by  $v_1, \dots, v_s, u_1, \dots, u_{n-1-s}$  where  $v_1, \dots, v_s$  parametrize  $Z$  around  $x$  identified with 0. Then slightly different from the earlier expression we have

$$\begin{aligned} \ell_p &= \sum_{j=1}^{n-1-s} \alpha_j u_j^2 + O(3), \\ g &= h(u) + \sum_{i=1}^s \gamma_i v_i + \sum_{i=1}^s \delta_i v_i^2 + O(3) \end{aligned}$$

where at least one of  $\gamma_i$  is nonzero since  $p$  is a nondegenerate point of  $g$  on  $Z$ . Once more by setting  $\partial(\ell_p + g)/\partial u_j$  equal to zero we see that  $u_1, \dots, u_{n-1-s}$  are all functions of  $v_1, \dots, v_s$ . On the other hand, we may assume none of the  $\delta_i$  are zero. For, suppose  $\gamma_1 \neq 0$  and some  $\delta_j = 0$ . Then replacing  $v_1$  by  $v_1 + v_j^2$  and keeping all other variables unchanged will result in a nonzero coefficient for  $v_j^2$  with all other  $\delta_i$  unchanged. Then setting  $\partial(\ell_p + g)/\partial v_i$  equal to zero yields

$$0 = \gamma_i + 2\delta_i v_i + O(2) =: H_i, \quad 1 \leq i \leq s.$$

As before, we see

$$\partial(H_1, \dots, H_s)/\partial(v_1, \dots, v_s) \neq 0.$$

Therefore, the implicit function theorem implies that there is only a single point solution, which is a nondegenerate critical point of  $\ell_p + g$  in a small neighborhood of  $x$  in  $W$ , which we can thus ignore.

Recall the local finiteness property in [4] that holds the key for proving that a taut hypersurface is real algebraic. We denote by  $\mathcal{G}$  the subset of  $M$  where the multiplicities of principal values are locally constant, and by  $\mathcal{G}^c$  its complement in  $M$ .

**Definition 4.** A connected Dupin hypersurface  $M$  of  $S^n$  has the *local finiteness property* if there is a subset  $S \subset \mathcal{G}^c$ , closed in  $M$ , such that  $S$  disconnects  $M$  into only a finite number of connected components, and for each point  $x \in \mathcal{G}^c \setminus S$ , there is an open neighborhood  $O$  of  $x$  in  $M$  such that  $O \cap \mathcal{G}$  contains a finite number of connected open sets whose union is dense in  $O$ .

It suffices to establish that  $\mathcal{G}$  satisfies the local finiteness property for  $M$  to be real algebraic [4, Theorem 8]. We begin with a convenient lemma. Recall the global minimum or maximum level set of a height function on  $M$  is called a top set. Setting  $j = 0$  in (1.1) we see a top set is always connected.

**Lemma 5.** *Let  $T_i$  be a sequence of top sets of dimension  $l$  at  $q_i$  in the taut hypersurface  $M$ . Suppose  $T_i$  converge to a top set  $T$  of dimension  $m$  at  $p$ . Then  $H_l(T) \neq 0$ .*

*Proof.* First off, the top-dimensional homology of a top set of  $M$  is nonzero. This follows from the Poincare duality (with  $\mathbb{Z}_2$  coefficients) and that a top set is connected.

Now let  $W$  be a tubular neighborhood of the top set  $T$  at  $p$  so small that  $T$  is the only critical set in it. Let  $j$  be so large that a tubular neighborhood  $W_j$  of the top set  $T_j$ , containing only  $T_j$ , is brought to lie inside  $W$ . Then by (1.2)

$$H_k(T_j) \rightarrow H_k(T)$$

is an injection for all  $k$ . It follows that  $H_l(T)$  is nonzero by what is said in the preceding paragraph.  $\square$

Returning to establishing the local finiteness property, let  $S \subset \mathcal{G}^c$  be the set of points where the principal multiplicities are  $(1, \dim(M) - 1)$  or  $(\dim(M) - 1, 1)$ . The set is closed; or else a boundary point of which would assume the single principal multiplicity  $(\dim(M))$  so that  $M$  would be a sphere.  $S$  must be a subset of  $\mathcal{G}^c$ . This is because if multiplicities  $(1, \dim(M) - 1)$  exist on an connected open set  $O \subset \mathcal{G}$ , let  $p_i \in O$  be a sequence which converges to  $p$  on the boundary of  $O$ . The multiplicities at  $p$  must remain to be  $(1, \dim(M))$ , or else it

would drop to the single multiplicity  $(\dim(M))$ . On the other hand, there must be a sequence  $q_i$  of points converging to  $p$  with fixed multiplicities  $(\dots, l)$  where  $l < \dim(M) - 1$ . Therefore, on the one hand, the curvature surface  $S_i$  at  $p_i$  with principal multiplicity  $\dim(M) - 1$ , which is a top set sphere of dimension  $\dim(M) - 1$ , converges to the top set curvature surface at  $p$ , which is also a sphere  $S_p$  of dimension  $\dim(M) - 1$ . This is because each  $S_i$  is cut out from its curvature sphere by a unique hyperplane  $L_i$  in the ambient Euclidean space, so that the limiting hyperplane also cuts out a sphere, which is  $S_p$ , from the limiting curvature sphere. On the other hand, at  $q_i$  the curvature surface  $T_i$  with principal multiplicity  $l$  is a top set as well, and so by Lemma 5 the  $l$ -dimensional homology in  $S_p$  is nontrivial, which is absurd.

We next show that  $S$  disconnects  $M$  into only finitely many components. Recall the following definition in [4].

**Definition 6.** For each natural number  $m$  we define  $(U_m^*)^+$  to be the collection of all  $x \in M$  for which there is a  $t > 0$  such that  $(x, t)$  is a regular point of the normal exponential map

$$E : (x, t) \mapsto \cos(t)x + \sin(t)\mathbf{n}$$

and such that the spherical distance function  $d_y$ , where  $y = E(x, t)$ , has index  $m$  at  $x$ .

We showed in Corollary 20 of [4] that  $(U_m^*)^+$  has a finite number of connected components for all  $m$ .

*Remark 7.* The  $+$  sign in  $(U_m^*)^+$  is merely to indicate that we traverse in the positive  $\mathbf{n}$  direction, which we have agreed to undertake earlier.

Consider  $A_m := (U_m^*)^+$  for  $m = 1, \dots, \dim(M) - 1$ . Let  $B := \bigcup_{m=2}^{\dim(M)-1} A_m$  and  $A := A_1$ . Then it is readily checked that  $M = A \cup B$  and furthermore  $C := A \cap B$  is exactly  $A$  with points of multiplicities  $(1, \dim(M) - 1)$  removed. Therefore, the Mayer-Vietoris sequence

$$0 \rightarrow H^0(M) \rightarrow H^0(A) \oplus H^0(B) \rightarrow H^0(C) \rightarrow H^1(M) \rightarrow$$

establishes that  $C$  has finitely many components, which is what we are after.

Now let  $x \in \mathcal{G}^c \setminus S$  and let  $Z$  through  $x$  be a critical submanifold with focal point  $p$ . By the nature of  $S$  we know that

$$\dim(Z) \leq \dim(M) - 2;$$

in particular,  $Z$  does not disconnect  $M$ . From this point onward we diversify into two cases.

Case 1. None of the curvature spheres of  $\ell_p + g$  contain  $Z$ .

This means that  $Z$  is not a level set of  $g$  so that  $g$  cuts  $Z$  in proper taut submanifolds. Let  $I$  be the index range such that

$$(2.1) \quad p = f_a(x), \forall a \in I.$$

Let  $W$  be a neck of  $Z$ . Let  $O \subset W$  around  $x$  be an open ball. The set

$$\mathcal{F}_O := \cup_{a \in I} f_a(O)$$

is a connected set of focal points around the focal point  $p$ .

We pick the open ball  $O$  so small that any critical submanifold of  $\ell_p + g = \ell_q$ , for focal points  $q \in \mathcal{F}_O$ , lies completely in  $W$  when its intersection with  $O$  is not empty. (From now on we identify an element  $q$  in  $\mathcal{F}_O$  with the corresponding  $g$  interchangeably.) Proposition 3 ensures that these critical submanifolds of  $\ell_q$  on  $W$  are all graphs over the corresponding critical submanifolds  $Z_g$  that  $g$  cut out on  $Z$ .

Consider the incidence space  $\mathcal{I} \subset \mathcal{F}_O \times W \subset S^n \times S^n$  given by

$$\begin{aligned} \mathcal{I} := \{ (g, z) : z \in \text{a critical submanifolds of } \ell_p + g \text{ in } W, \\ \text{and } \dim(Z_h) \text{ is not locally constant for } h \text{ around } g \}. \end{aligned}$$

Let

$$\Pi : S^n \times S^n \rightarrow S^n$$

be the standard projection onto the second factor. Then

$$(2.2) \quad W \cap (\mathcal{G}^c \setminus Z) = \Pi(\mathcal{I}).$$

Note that  $\Pi|_{\mathcal{I}}$  is an open finite (hence proper) map; the finiteness is because through each point in  $M$  there are only at most  $\dim(M)$  worth of critical submanifolds, while the openness follows from that of  $\Pi$ .

The following lemma, based on our inductive hypothesis, makes the structure of  $\mathcal{I}$  clear.

**Lemma 8.**  *$\mathcal{I}$  is a piecewise smooth simplicial complex of dimension at most  $\dim(M) - 1$ .*

*Proof.* Since  $Z \subset S^n$  is algebraic by the inductive hypothesis, the set  $UN^o$  of unit normals  $\xi$  of  $Z$  at which the shape operator  $S_\xi$  has multiplicity change is semialgebraic. This can be seen as follows. Let  $\dim(M) = s$  and let  $(y, \zeta) \in Z \times S^{n-s-1}$  parametrize the unit normal bundle of  $Z$ . The characteristic polynomial of  $S_\xi$  is of the form

$$\lambda^s + a_{s-1}\lambda^{s-1} + \cdots + a_1\lambda + a_0,$$

where  $a_1, \dots, a_{s-1}$  are polynomials in the zero jet of  $\zeta$  and the second jets of  $y$ ; hence they are Nash functions. By the slicing theorem [2, p. 30],  $Z \times S^{n-l-1}$  is decomposed into finitely many disjoint semialgebraic



sets  $A_1, \dots, A_m$ , where each  $A_i$  is equipped with semialgebraic functions  $\eta_{i_1} < \dots < \eta_{i_i}$  that solve the characteristic polynomial. Where multiplicities are not locally constant occurs at some  $A_1, \dots, A_m$  whose dimensions are lower than  $n - 1$ , the dimension of  $Z \times S^{n-1-s}$ .

Now in view of Corollary 2, for a unit normal  $\xi$  to  $Z$ , we let  $q_\xi^1, q_\xi^2, \dots$ , and  $q_\xi^{\dim(Z)}$  be the focal point of the curvature surface through the base point of  $\xi$  corresponding to the principal curvature function  $\lambda^1(\xi), \dots$ , and  $\lambda^{\dim(Z)}(\xi)$  of  $S_\xi$ , respectively. The remark following Corollary 2 gives the focal maps  $g^1, \dots, g^{\dim(Z)}$  that send  $\xi$  to the respective focal points; by the algebraic nature of  $Z$ , all these maps are semialgebraic. Consider the semialgebraic set  $\mathcal{X} \subset UN^o \times S^n \times S^n$  defined by

$$\mathcal{X} := \{(\xi, q, r) : q = g^j(\xi) \text{ for some } j; r \text{ belongs a critical set of } Z \text{ of the height function } \ell_q \text{ centered at } q\}.$$

Due to the nature of all these defining functions,  $\mathcal{X}$  is semialgebraic. (For instance, critical submanifolds are obtained by setting the first derivative of the height function equal to zero on  $Z$ , which is a semialgebraic process.) Let  $pr : UN^o \times S^n \times S^n \rightarrow S^n \times S^n$  be the standard projection, and let  $\mathcal{J} := pr(\mathcal{X})$ . The set  $\mathcal{J}$  is also semialgebraic.

We now estimate the dimension of  $\mathcal{J}$ . Consider the the map

$$PR := \Pi|_{\mathcal{J}}.$$

It is readily seen that  $PR : \mathcal{J} \rightarrow Z$ . For a fixed  $z$  in the image of  $PR$ , the preimage  $PR^{-1}(z)$  consists of the focal points that come from the  $\xi \in UN^o$  where the base point of  $\xi$  is  $z$ . At  $z$ , the eigenvalue problem is an algebraic one; therefore, the set  $S$  of  $\xi$  based at  $z$  where principal multiplicities is not locally constant is a subvariety of the unit normal sphere at  $z$  of dimension at most  $n - \dim(Z) - 2$ . Each  $\xi$  in  $S$  gives rise to at most  $\dim(Z)$  worth of taut submanifolds through  $z$ , and vice versa, whose focal points are the ones in  $PR^{-1}(z)$ . Therefore,

$$\dim(PR^{-1}(z)) \leq n - \dim(Z) - 2.$$

As a result, as  $z$  varies in  $Z$

$$\dim(\mathcal{J}) \leq n - \dim(Z) - 2 + \dim(Z) = \dim(M) - 1.$$

Since a semialgebraic set assumes a triangulation of semialgebraic simplicial complexes [2, p. 217], the structure of  $\mathcal{J}$  is clear. Consider the map  $F : \mathcal{I} \rightarrow \mathcal{J}$  given by

$$F : (g, z) \rightarrow (g, \pi(z)),$$

where  $\pi$  is given in Proposition 3. The preimage of each point is finite with cardinality at most  $\beta(M, \mathbb{Z}_2)$  between the two spaces with the

naturally induced metrics. Hence,  $F$  is a finite covering map, since for a fixed  $g$  the map  $\pi$  maps a critical manifold of  $\ell_p + g$  to  $Z_g$  diffeomorphically. As a consequence  $\mathcal{I}$  inherits from  $\mathcal{J}$  a piecewise smooth triangulation of dimension  $\dim(M) - 1$  sitting in  $S^n \times S^n$ . In fact we can work our way down the skeletons of the simplicial complex dimension by dimension. Each open face of the skeleton is defined by a finite set of polynomial functions  $H < 0$ , so that the pullback maps  $H \circ F < 0$  define the corresponding open face for  $\mathcal{I}$ .  $\square$

Since the natural projection  $\Pi : S^n \times S^n \rightarrow S^n$  into the second slot is an open finite map when restricted to  $\mathcal{I}$  as mentioned earlier, we see that at  $x \in Z$  with preimages  $x_1, \dots, x_k \in \mathcal{I}$ , the projection  $\Pi$  sends  $k$  disjoint piecewise smooth (local) finite simplicial complexes  $\mathcal{C}_1, \dots, \mathcal{C}_k$  (of dimension at most  $\dim(M) - 1$ ) around  $x_1, \dots, x_k$ , respectively, to  $x \in S^n$ . Over each  $\mathcal{C}_j$ , the differential  $d\Pi$  is not defined over the skeletons of dimension  $\leq \dim(M) - 2$ ; call this set  $\mathcal{K}_j$ , which is a rectifiable set [6, p. 251]. Hence by the general area-coarea formula [6, p. 258]

$$\mathcal{H}.\dim(\Pi(\mathcal{K}_j)) \leq \dim(M) - 2$$

since  $\Pi|_{\mathcal{I}}$  is a finite map; here  $\mathcal{H}.\dim$  denotes the Hausdorff dimension. On the other hand,  $d\Pi$  is defined over the  $(\dim(M) - 1)$ -dimensional open faces  $\mathcal{F}_{jl}$  of  $\mathcal{C}_j$ . By Federer's version of Sard's theorem [6, p. 316], the critical value set  $\Theta_{jl}$  of  $\Pi$  over  $\mathcal{F}_{jl}$  satisfies

$$\mathcal{H}^{\dim(M)-1}(\Theta_{jl}) = 0,$$

where  $\mathcal{H}^\nu$  denotes the Hausdorff  $\nu$ -dimensional measure. Therefore,

$$\mathcal{H}^{\dim(M)-1}(\Pi(K_j \cup_l \mathcal{F}_{jl})) = 0,$$

which implies that  $\Pi(\mathcal{K}_j \cup_l \mathcal{F}_{jl})$  does not disconnect  $M$  [11, p. 269]. Over the regular points of  $\mathcal{F}_{jl}$  the map  $\Pi|_{\mathcal{I}}$  is a finite covering map. As a consequence  $\Pi$  maps  $\mathcal{I}$  to  $k$  "folded" local complexes around  $x$ , each locally disconnects  $M$  in only finitely many components.

Case 2. There are some  $g$  such that  $\ell_p + g$  contain  $Z$ .

This means  $Z$  is contained in a level set for such  $g$ . Suppose  $Z$  is contained in a critical submanifold of  $g$ . Then by Corollary 2, the height functions  $\ell_p$  and  $\ell_p + g = \ell_q$  share the same center of the curvature sphere through  $Z$ , so that it must be that  $p = q$ , which is not the case. Therefore, all points of  $Z$  are regular points of  $g$ . Similar to the equations following Proposition 3 we have

$$\begin{aligned}
(2.3) \quad \ell_p &= \sum_{j=1}^{n-1-s} \alpha_j u_j^2 + O(3), \\
g &= h(u) + O(3);
\end{aligned}$$

That  $g$  has no  $v$  terms is because  $g(Z)$  is a constant. Analogous analysis as before shows that  $u_1, \dots, u_{n-1-s}$  are functions of  $v_1, \dots, v_s$ , so that the critical manifolds of  $\ell + g$  are graphs over  $Z$ .

In fact, we can understand all these  $g$  explicitly. Let  $S^l$  be the smallest sphere containing  $Z$ . It is more convenient to view what goes on in  $\mathbb{R}^n$  when we place the pole of the stereographic projection on the  $S^l$  containing  $Z$ . Then we are looking at an  $\mathbb{R}^l$ , which we may assume is the standard one contained in  $\mathbb{R}^n$ , in which  $Z$  sits. Let  $E \simeq \mathbb{R}^{n-l}$  be the orthogonal complement of the  $\mathbb{R}^l$ . Any  $\mathbb{R}^{n-l-1}$  in  $E$  gives rise to an  $\mathbb{R}^{n-1}$  containing  $Z$ , and vice versa. Back on the sphere, this means that we have an  $(n-l-1)$ -parameter family of  $S^{n-1}$  containing  $Z$ . The focal points of these  $S^{n-1}$  is an  $S^{n-l-1}$  on the equator. Now the critical sets of this  $(n-l-1)$ -parameter family of distance functions centered at the focal sphere  $S^{n-l-1}$  are all graphs over  $Z$  by the analysis following (2.3). It follows that we have a manifold structure  $Z \times S^{n-l-1}$  of dimension

$$\dim(Z) + n - l - 1 \leq n - 2 = \dim(M) - 1$$

if  $\dim(Z) < l$ , in which case, the set of all these critical submanifolds locally disconnects  $M$  in at most two components. If on the other hand  $\dim(Z) = l$ , then  $Z = S^l$ . The manifold structure  $Z \times S^{n-l-1}$  of dimension  $n$  then fills up  $M$ , which means there is no multiplicity change around  $Z$  so that  $Z$  can be ignored.

In summary, we have established the local finiteness property, and so  $M$  is algebraic when it is a hypersurface.

We now handle the case when  $M$  is a taut submanifold. It is more convenient to work in  $\mathbb{R}^n$ . Let  $M_\epsilon$  be a tube over  $M$  of sufficiently small radius that  $M_\epsilon$  is an embedded hypersurface in  $\mathbb{R}^n$ . Then  $M_\epsilon$  is a taut hypersurface [10], so that by the above  $M_\epsilon$  is algebraic. Consider the focal map  $F_\epsilon : M_\epsilon \rightarrow M \subset \mathbb{R}^n$  given by

$$(2.4) \quad F_\epsilon(x) = x - \epsilon \xi,$$

where  $\xi$  is the outward field of unit normals to the tube  $M_\epsilon$ . Any point of  $M_\epsilon$  has an open neighborhood  $U$  parametrized by an analytic algebraic map. The first derivatives of this parametrization are also analytic algebraic [2, p. 54], and thus the Gram-Schmidt process applied to these first derivatives and some constant non-tangential vector

produces the vector field  $\xi$  and shows that  $\xi$  is analytic algebraic on  $U$ . Hence  $F_\epsilon$  is analytic algebraic on  $U$  and so the image  $F_\epsilon(U) \subset M$  is a semialgebraic subset of  $\mathbb{R}^n$ . Covering  $M_\epsilon$  by finitely many sets of this form  $U$ , we see that  $M$ , being the union of their images under  $F_\epsilon$ , is a semialgebraic subset of  $\mathbb{R}^n$ . Then the Zariski closure  $\overline{M}^{\text{zar}}$  of  $M$  is an irreducible algebraic variety of the same dimension as  $M$  and contains  $M$ .

The inductive procedure is thus completed.

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